Analyzing Errors and Misconceptions of 11th Grade Learners in Solving Tangent-Chord Theorem Problems: A Case in Tshwane North District Secondary School

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ABSTRACT

This qualitative case study, grounded within the interpretive paradigm, analyzed the errors and misconceptions made by 11th-grade learners when tackling the tangent-chord theorem task in Euclidean geometry. Studying Euclidean geometry helps learners develop critical thinking skills, such as constructing arguments and applying logical reasoning. Analyzing facts and diagrams when addressing Euclidean geometry issues helps learners identify appropriate theorems. It focused on exploring, describing, and explaining errors and misconceptions based on Van Hiele's theory, which was used to understand the geometric reasoning levels of the learners. The study was conducted in a township public secondary school in the Tshwane North District of Gauteng, South Africa, involving 30 Grade 11 mathematics learners as participants. The finding reveals that most learners operated at or below Van Hiele levels 1 and 2, relying primarily on visual cues and memorized procedures rather than conceptual understanding. Errors and misconceptions arose due to the learners' incorrect angle labeling, flawed assumptions, poor diagram interpretation, and misuse of geometric terminology. Notably, 16,7% of learners showed no understanding of the concept. While 36,7% of learners made repeated statement errors, highlighting systematic challenges in visualization and reasoning. These misconceptions were found to be linked to instructional gaps, overgeneralization of geometric rules, and limited language precision. In response, the study suggests that teachers integrate dynamic visualization tools such as GeoGebra, embed open-ended conceptual tasks, promote collaborative peer learning, and contextualize geometry through real-world applications. These strategies aim to deepen learners' conceptual understanding, strengthen spatial reasoning, and support progression through the Van Hiele levels of geometric thinking.

Keywords: Errors; misconceptions; Euclidean geometry; geometric reasoning; tangent-chord theorem

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INTRODUCTION

Many studies (Makhubele, Nkhoma, & Luneta, 2015; Masilo, 2018; Machisi, 2021; Mutambara & Bansilal, 2022; Mosia, Matabane, & Moloi, 2023) have identified Euclidean Geometry as a challenging subject for many learners; however, it is also the most practical and applicable form of mathematics at all educational levels. It also offers learners an opportunity to develop argumentation skills and enhance their inductive and deductive reasoning abilities. In South Africa, statistical data from the Department of Basic Education (2021; 2022; 2023) indicates a concerning trend of poor performance in geometry among learners. The Department of Basic Education's diagnostic analyses (2021; 2022; 2023) indicate that learners are struggling in mathematics, with Euclidean geometry being one of the subjects in which their performance is particularly poor. Despite the practical applications of Euclidean geometry, learners consistently face substantial challenges, evident in their performance on national assessments. This study focuses on the tangent-chord theorem as a lens through which persistent misconceptions can be analysed and addressed, given their relevance in geometry, other topics such as trigonometry in 2D and 3D, coordinate geometry etc., and broader mathematical problem-solving.

The analysis of errors and misunderstandings in learners' mathematical development, particularly in the context of geometry learning, has attracted considerable attention. Several studies have detailed prevalent misconceptions and errors made by learners in geometry (Biber et al., 2013; Makhubele et al., 2015; Mohyuddin & Khalil, 2016; Dwijayanti et al., 2018; Ayuningsih et al., 2020), highlighting the significance of research in this area. Teachers need to be capable of analyzing errors, predicting common misconceptions, and interpreting learners' incomplete thoughts to effectively aid learners' learning (Makhubele et al., 2015). The frequency of errors among learners, as indicated by the mistakes present in their work, underscores the necessity for teachers to take proactive measures to address these challenges (Ayuningsih et al., 2020). This is a very crucial part for teachers to deal with when planning the lesson. This indicates that the majority of teachers overlooked the issue of learners' errors and misconceptions. Making mistakes is a natural part of the learning process, but frequent and significant errors require attention and follow-up to prevent adverse effects on learners.

Misconceptions acquired at one educational level can persist at higher levels of education (Khasanah et al., 2020). Dwijayanti et al. (2022) elaborate on the causes of errors in problem-solving, which may include learners' lack of understanding of the subject matter or prerequisite material, inadequate mastery of language or mathematical symbols, misinterpretation or misuse of formulas, errors or lack of thoroughness in calculations, forgetfulness of concepts, inadequate support from teachers in understanding the material or concepts being taught, and teachers paying less attention to learners during the learning process. These can hinder the learning process of mathematics since it is progressive.

The Van Hiele model of geometric thinking is a progressive and comprehensive framework used to understand how learners learn geometry. The Van Hiele Theory is widely recognized for its effectiveness in guiding teachers to facilitate the learning of geometry in a structured and meaningful manner (Luneta, 2014; Fitriyani et al., 2018; Amidu & Nyarko, 2019). This theory comprises two main components: the "levels of thinking'' and the "phases of learning" (Howse & Howse, 2015). The levels of thinking represent the stages of understanding that learners progress through as a result of instruction, and they include visualization, analysis, informal deduction, deduction, and rigor. These levels are sequential and hierarchical, and learners advance through them as they gain deeper insights into geometry (Vojkuvkova, 2012; Amidu & Nyarko, 2019; Chiphambo & Feza, 2020). Furthermore, the five phases of learning within the Van Hiele model are information, directed orientation, explication, free orientation, and integration. These phases provide a structured approach to guide learners in their geometric learning journey. The Van Hiele theory, as noted by Luneta (2014), Robichaux-Davis & Guarino (2016), and Nisawa (2018), is not dependent on learners' age or their progression from one level to the next but rather on the quality of teaching and the learning opportunities they receive. In summary, using the Van Hiele theory is essential for learners to succeed in learning geometry, providing a structured approach to teaching this fundamental mathematics subject.

LITERATURE REVIEW

Euclidean Geometry

Geometry involves learning about the geometric properties of a figure that do not change when it is revolved or transformed (Mahlaba & Mudaly, 2022). These properties include points, lines, planes, angles, different shapes, and dimensions. Geometry is an important discipline of mathematics and has been acknowledged as a domain that can enliven mathematics (Machisi, 2021). Geometry appeals to our different (visual, instinctive, and aesthetic) senses and captures the attention of learners when we use shapes and constructions that learners may relate to within their real-world contexts (Naidoo & Kapofu, 2020). According to Maweya and Pule (2024), these can be applied to a variety of real-world triangular shapes, including buildings, homes, flagpoles, maps, trees, hair braiding styles, and any triangular arrangement of people or objects. Working with geometric constructions enhances learners' thinking abilities (Mahlaba & Mudaly, 2022), a notion that the current South African mathematics curriculum supports. The South African Curriculum Assessment and Policy Statements (CAPS) capture the notion that learners of mathematics must be able to question, examine, conjecture, and experiment (Department of Basic Education, 2014). The skills necessary to examine, question, conjecture, and experiment may be acquired through learning geometry. These attributes promote logical thinking in learners and engage them in analytical and rigorous thinking.

In the study of Kpotosu, Amegbor, Mifetu, and Ezah (2024) they investigated Senior High School learners' difficulties with Geometry topics and found that 156 learners (52.0%) found circle theorems challenging, while 138 learners (46.0%) found it difficult to understand the relationship between a circle's radius and tangent. Angle formed by a tangent and a chord: 146 learners, or 48.7%, thought it was challenging. External tangents: 141 learners (47.0%) thought it was challenging. In this study, we closely examine the specific challenges that learners encounter when engaging with the tangent-chord theorem task.

Errors and misconceptions

In mathematics, learners must go through several steps to arrive at their ultimate solution while solving mathematical questions. The steps involved in solving a mathematical question are as follows: a) comprehend the problem, b) come up with a plan, c) execute the plan, and d) review the result (Aidoo-Bervell, 2021). Learners may easily answer any mathematical issue with the aid of these procedures. However, due to differences in their thought processes and challenges in comprehending teaching methods, some pupils struggle to answer mathematical problems. Learners make a variety of assumptions, errors, and misconceptions as a result. Errors are deviations from what is generally considered correct. Themane and Luneta (2021) describe the error as "basic indications of the problems a learner is having throughout a learning session". Errors are viewed as evidence gathered from anything that is not quite correct after solving a problem using an unsuitable method (Aidoo-Bervell, 2021). For instance, (angle, angle, angle) similarity asserts that two triangles with three congruent angles are comparable. However, they are not always consistent, so the pupil has made an error.

The errors that occur in a process can be classified into two main categories: systematic and non-systematic errors. Systematic errors are deliberate and occur when incorrect responses are consistently given. These inaccuracies can be challenging to detect because the same incorrect responses may appear multiple times. Systematic errors can become deeply ingrained, and it is difficult for learners to correct them unless they receive assistance in identifying and addressing their mistakes. On the other hand, non-systematic errors are inadvertent and do not occur repeatedly, especially in calculations, resulting in incorrect solutions. These errors often stem from carelessness and can be more easily rectified by learners themselves (Aidoo-Bervell, 2021). According to Motseki and Luneta (2024), common errors include systematic mistakes, random errors, and careless oversight. Systematic errors stem from a misunderstanding or lack of comprehension of ideas and principles. Random errors are not easily linked to specific difficulties. Careless errors may be unintentional, such as forgetting to include a sign (Quinio & Cuarto, 2023). Errors often result from mistakes and carelessness, while misconceptions arise from misunderstandings, highlighting the importance of distinguishing between the two. Recent empirical research has delved into how learners develop these beliefs, categorizing them as either systematic, random, or thoughtless mistakes (Makonye & Khanyile, 2015). Systemic errors arise from misconceptions of fundamental ideas, while random errors lack discernible patterns (Themane & Luneta, 2021).

Misconceptions in learner learning are ingrained errors that are hard to spot because they are made unintentionally. They occur when learners misunderstand a topic based on their ideas, theories, explanations, or understandings. According to Themane and Luneta (2021), a misconception is a learner's belief that leads to a series of errors due to the incorrect application of a rule, over- or under-generalization, or a different interpretation of the situation. This can stem from a flawed cognitive framework that supports, explains, or justifies the error.

THEORETICAL FRAMEWORK

This study draws on the Van Hiele theory of geometric thinking, which posits five ordered levels of thinking, namely, visualization, analysis, informal deduction, formal deduction, and rigor (Van Hiele, 1986). These are levels of mathematical sophistication, meaning that you need to master one level before moving on to the next one. For example, for secondary school geometry, this theory is useful in diagnosing where gaps in learners' reasoning might have occurred to inform the design of instructional strategies.

In this study, the focus is on the first three levels, viz, visualization, analysis, and informal deduction, which play a pivotal role in building an understanding of the tangent-chord theorem. Visualization involves recognizing geometric figures based on their overall appearance, often without considering their properties, regardless of the position or print of the figure (Armah & Kissi, 2019). At the analysis level, learners start to notice and define the characteristics of geometric figures. They begin to progress from recognizing shapes to understanding what makes those shapes unique. Informal deduction refers to understanding relationships between properties and being able to provide logical arguments within the limits of a given problem (Van Hiele, 1986).

The Van Hiele framework also encapsulates learning stages like information, guided orientation, explication, free orientation, and integration. These phases also provide a pedagogical roadmap, stressing the necessity of providing structured learning activities that guide learners through the stages of reasoning (Howse & Howse, 2015; Alex & Mammen, 2018). The illustrations, such as guided or hands-on orientations, include tasks in which learners can manipulate and explore geometric figures as a strategy to gain a deeper understanding of the mathematical content.

In this study, tasks were designed to reveal learners' reasoning on different levels and to uncover misconceptions that inhibit learners from understanding the tangent-chord theorem. The study maps these learners' errors and misconceptions to the Van Hiele levels and provides insights into instructional strategies tailored to address the reasoning gaps. Previous research emphasizes the diagnostic and prescriptive power of the Van Hiele framework as a valuable approach to addressing learners' difficulties in geometry.

RESEARCH METHODOLOGY

This paper employed a qualitative case study design, rooted within an interpretive paradigm, to explore Grade 11 learners' conceptual understanding and misconceptions regarding the tangent-chord theorem. The interpretive paradigm was appropriate due to its focus on learners' subjective experiences and reasoning process in geometry. Günbayi and Sorm (2018) describe how this paradigm is structured to capture the nuances of people's experiences and how they interpret them. The interpretive paradigm aligns closely with Van Hiele's model of geometric thinking, which emphasizes instructional quality over age in shaping learners' geometric development (Crowley, 1987).

A purposive sampling strategy was used to select the participants and research site. The study was conducted in a township public secondary school in the Tshwane North District of Gauteng, South Africa. A total of 30 11th-grade mathematics learners participated in the main study. These learners were selected based on their enrolment in mathematics and availability during scheduled teaching time. To validate the tangent-chord geometry task and instructional clarity, a pilot study was conducted with 10 grade 11 learners from a different school within the same district. These learners were not included in the main study's analysis. This pilot study helped to refine both the item design and the instructional wording, ensuring alignment with Van Hiele's theoretical expectations.

The data was collected using a two-item task that focuses on the tangent-chord theorem. Each item was mapped to specific Van Hiele levels, particularly levels 1, 2, and 3, which are visualization, analysis, and informal deduction, respectively. Learners completed the tasks in an invigilated setting during normal classroom time. Responses were written, collected, and subsequently transcribed for qualitative analysis.

Reliability and Validity

The creation of the tangent-chord theorem tasks underwent a careful validation process to ensure that they were aligned with the research aims and theoretical framework. The task set was read by three leading experts in mathematics education who had a proven history of designing both instruction and assessment in geometry. These experts reviewed the tasks for coherence with curriculum goals and to what extent the tasks were able to assess reasoning at different Van Hiele levels. The tasks were mapped to curriculum standards and learning goals from secondary-level mathematics education to establish content validity. Construct validity was established by ensuring that the tasks elicited responses (Du Plooy-Cilliers et al., 2021) that could be mapped onto specific Van Hiele levels of reasoning, ranging from visualization to formal deduction.

A thematic analysis approach was adopted to interpret the data guidance of the Van Hiele framework, and qualitative data from learners' responses were analysed. The coding was done in the following steps:

Step 1: Familiarization with the Data: All learner responses were transcribed, read, and reread through for immersion in the data.

Step 2: Code Generation: An initial code set was developed based on the Van Hiele framework (e.g., “visualization errors,” “reasoning gaps,” “conceptual misunderstanding”).

Step 3: Iterative Refinement: Inductive development of further codes as patterns emerged in the data. Specific themes, such as errors arising from misapplying tangent-chord relationships or not realizing angles are congruent. Various themes were then created with specific Van Hiele levels

Step 4: Inter-Rater Reliability: A subsample of the data was coded independently by two coders to assess consistency in the application of codes. Disagreements were resolved via discussion, and a final coding scheme was developed.

Categorizations and Interpretations: Codes were brought together into themes that guided (some) overarching directions (i.e., lack of ‘visualization,’ ‘over-reliance’ on procedures, and ‘misconceptions’ conceptually) and returned to the Van Hiele levels in an attempt to glean insight into learners’ geometric reasoning.

FINDINGS

Overall Performance across the Items

In the tangent-chord task, participants were given two items to complete (see Figure 1). Table 1 provides details about the items, the Van Hiele level they targeted, and the learners’ performance on each item. The table shows that the participants generally demonstrated greater proficiency in determining the size of angle y compared to angle x. While they found it relatively straightforward to calculate the numerical value of the angle, articulating the reasoning behind their responses proved to be a significant challenge for them.

Figure 2 below represents a stacked column indicating correct and incorrect values and reasoning using different colors, where orange is for the correct value of x, yellow is for the correct reason of x, green is for the correct value of y, and brown is for correct reason of y value, and there is bar it indicates that the learner(s) got incorrect values and reasoning. The data presented in Figure 2 revealed that out of the 30 participants, only eight learners were able to provide the correct answer by citing “tangent-chord” as their reasoning. These eight learners demonstrated a high level of attentiveness and analytical thinking, enabling them to apply the relevant theorem to the given problem. The capacity to discern and derive the characteristics of a geometric object by leveraging its intrinsic properties, connections with other objects/diagrams, and governing principles is widely recognized as a fundamental aspect of a successful proof process, as emphasized by Noviana and Hadi (2021). Additionally, it is noteworthy that five of the thirty participants provided incorrect answers, highlighting the need for further examination of the underlying conceptual challenges.

1. Consider the diagram below and answer the following questions given that the lines ABT and EB meet at B.

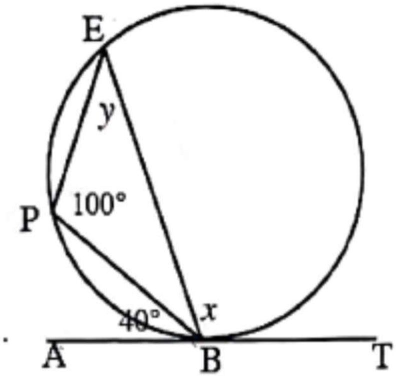


Figure 1: *The tangent-chord task*

Table 1: *Question items with VH level and results per question*

Item Number	Question items	Target as per VH level	Number of learners who obtained correct answers per item.
a)	What is the size of the angle x?	VHL2 Know the tangent-chord theorem	19
	Give reasons and show your work.	VHL3 Know geometric reasoning	12
b)	What is the size of the angle y?	VHL2 Know the tangent-chord theorem	25
	Explain your answer with detailed reasoning.	VHL3 Know geometric reasoning	16

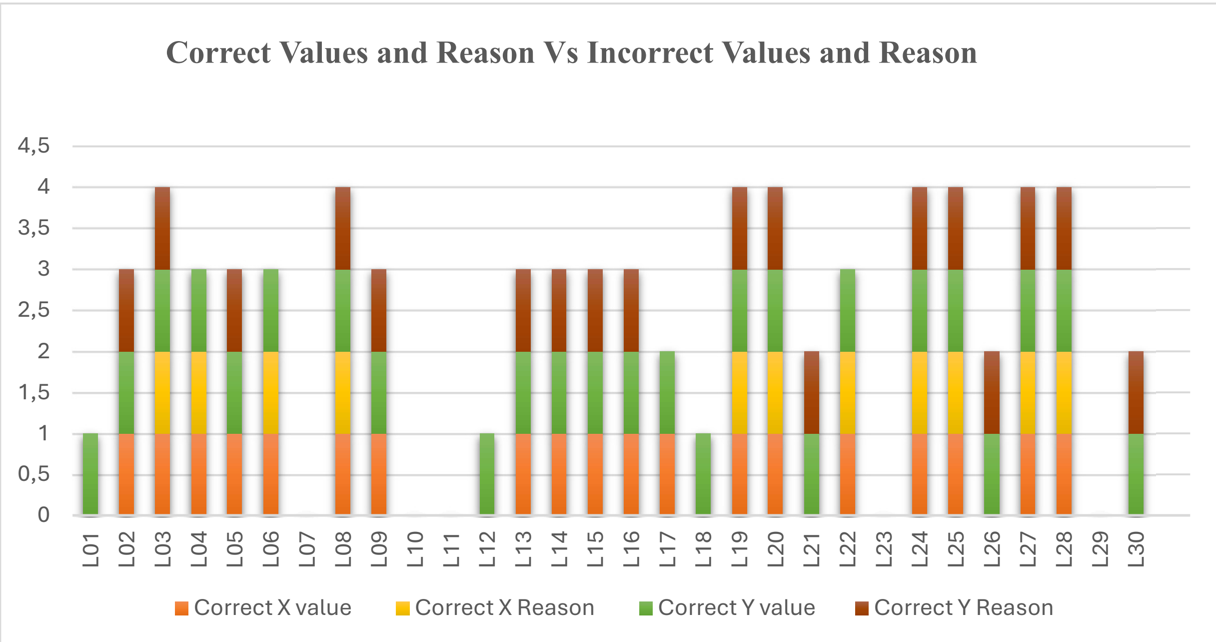


Figure 2: *Learner performance of the tangent-chord task*

In the research results, Table 1 and Figure 2 are supported by the data presented in Tables 2 and 3. Table 1 indicates that 11 learners made errors in their statements regarding Item A. For example, in Table 2, learner L13 incorrectly labelled angle ∠EBP as "$\angle AB_2P$," demonstrating a lack of understanding in naming the angle and suggesting a level below level 1 of Van Hiele's geometric levels. Furthermore, out of 30 learners, 18 were unable to provide correct reasoning for the given statement, indicating reasoning errors. For instance, learner L02 supported the statement (x = P) with "Ext ∠," which is a reasoning error as x is not an exterior angle of triangle BPE. Additionally, learners L09 and L14 stated "∠ between a line and a chord" without specifying the line, representing a reasoning error. Finally, learner L15's statement "x = 100 (∠ on a str. line)" indicates a reasoning error due to the misunderstanding that there is not just one angle on a straight line but rather more than one adjacent angle. In Table 1, it is observed that 11 out of 30 learners made both statement and reasoning errors. Table 2 provides examples of such learners: L01, L05, L07, L10, L11, and L12. This subset of learners is then used to analyze the findings further.

Analysis of the Tangent-Chord Concept of L07

Understanding the Tangent-chord Theorem

The Tangent-chord theorem states that when a tangent touches a circle at point T and a chord AB is drawn through it, then the angle ∠ATB is equal to the angle ∠ACB, where C is any point on the circle. This theorem showcases the fundamental relationship between tangents and chords, which is essential knowledge for secondary school learners.

The Role of Assumptions

The frequent presence of assumptions in L07 statements, for example, assuming that $\angle B_1 = 50$ and using the tangent-chord theorem as support, indicates that the learner is depending on intuition rather than formal proofs. This demonstrates a key aspect of the van Hiele levels, as learners at lower levels often fail to employ formal deductive reasoning or grasp the importance of proof in geometry.

Learners' errors

The response from learner L07 contained errors in both the statement and the reasoning. The learner incorrectly stated that the value of angle ∠B was 50^0 and attributed it to "a tan chord." These errors led to misconceptions being formed, as other learners used the incorrect information to proceed.

Analysis of the Tangent-chord Concept of L10

The response provided by learner L10, claiming that the angle $\angle B = 90^0$ is due to "a tan perpendicular to a chord," demonstrates a significant misunderstanding of key geometric principles, particularly the tangent-chord theorem and related concepts. This examination will scrutinize the implications of the response and elucidate how it reveals broader issues in geometric comprehension.

Misapplication of Geometric Principles

Erroneous Reasoning: The phrase "a tan perpendicular chord" reveals a fundamental misconception regarding how tangents and chords interact in a circle. In Euclidean geometry, it is well-established that tangents are not described as "perpendicular" to chords. Instead, it is appropriate to say that lines can be perpendicular to chords or that radii or diameters can be perpendicular to tangents. The tangent to a circle forms a right angle with the radius at the point of tangency, which is a crucial aspect of the tangent-chord theorem. Learner L10 appears to conflate these concepts, indicating a lack of clarity concerning the definitions and relationships of geometric elements.

Visualization and Diagram Interpretation

Poor Visualization Skills: Learner L10's response demonstrates an evident inability to carefully consider the diagram. Proficient visualization is crucial in geometry, enabling learners to comprehend the relationships between different elements within a circle. Without this skill, learners may misinterpret the roles of various lines and angles, leading to errors. For instance, in the context of the tangent-chord theorem, understanding that the angle formed between a tangent and a chord is equal to the angle in the opposite segment is essential. The learner's failure to recognize this relationship suggests a need for enhanced instruction focused on developing visualization skills.

Understanding Geometric Relationships: In geometry, comprehending how various elements interact is vital. For example, while the tangent to a circle does form a right angle with the radius at the point of tangency, the angle formed between the tangent and a chord is not directly related to being perpendicular. Instead, learners should grasp that this angle relates to the arcs and angles opposite to it. This nuanced understanding is crucial for correctly applying the tangent-chord theorem.

Learners' Errors

Learner L10's response, which asserted that the value of angle $\angle B = 90^0$ and attributed it to "a tan perpendicular to a chord," exhibited both statement and reasoning errors. Furthermore, the learner's utilization of these errors led to the formation of misconceptions.

Table 1 shows that five learners had difficulty determining the correct value of y in Item B. This suggests that these learners had difficulty in finding the numerical value of an angle, indicating that they are operating at a level below Van Hiele's geometric level 1. Moreover, 14 out of 30 learners were unable to provide a correct reason for their statements, indicating errors in reasoning. For example, learner L17 incorrectly used the term "chord subtended" to justify $y = 40^0$, which is not a valid geometric reasoning. Additionally, learners L18 and L22 did not attempt to provide a reason at all. Furthermore, in Table 1, 5 out of 30 learners made errors in both their statements and their reasoning. Table 2 presents some of these learners, including L07, L10, L11, L23, and L29. Although only the response of L29 is shown, the following analysis is discussed based on L17.

Analysis of L17's Response

Correct Calculation but Inappropriate Reasoning

The learner correctly identified ($x = 100^0$), showing some understanding and ability to calculate the value. However, the reasoning provided was related to cyclic quadrilaterals, which is not relevant to the problem. This indicates a discrepancy between the learner's ability to follow procedures correctly and their understanding of the underlying concepts. In geometry, especially at advanced levels, correct answers should be accompanied by appropriate reasoning. Additionally, the learner accurately determined the numerical value ($y = 40^0$), also indicating a certain level of understanding. However, an error in reasoning was made when supporting this statement.

Misapplication of Theorems

It seems that L17 may be familiar with the concept of "opposite angles of a cyclic quadrilateral" but may not realize that it does not apply to the current problem. This indicates that L17 might be relying on memorization rather than a deep understanding of the relationships involved. In the case of the tangent-chord theorem, it would have been better for the student to understand the importance of ABT (tangent) and the circle's angle relationships.

Lack of Geometric Insight

The phrase "sorry I didn't use tangent-chord" indicates uncertainty and a lack of confidence. It implies that L17 is aware of other relevant concepts but feels unable to connect them to the problem at hand. This is a critical point; while L17 has a correct answer, the underlying reasoning does not demonstrate a robust understanding of the tangent-chord theorem or how it applies to the problem.

Table 2: *Question item A shows that learners from L01 to L15 responded with errors.*

Learner codes	Question Item A	Researchers' comments
L01	$x = 100 - 40 = 60°$ tan chord AP is a chord PB = chord ABT is a tangent EB Bisect the chord at B EPB is a segment	Calculation/ Statement error AP is a chord (Statement error and operating below VHL1) EB bisects the chord (Assumption Error)
L02	~~$\hat{E} = 40°$~~ ~~$\hat{B}_1 = x = 40°$~~ $\hat{B}_2 + \hat{P} + \hat{E} = 180°$ $\hat{B}_1 = 180° - 100° - 40°$ $\hat{B}_1 = 40°$ $\hat{B}_1 + \hat{B}_x + \hat{B}_2 = 180°$ Str. line or $x = P$.... Ext ∠ $\hat{B}x = 180° - 40° - 40°$ $x = 100°$ $\hat{B}x = 100$	Reasoning error
L05	$P\hat{B}A = P\hat{B}E$ ($x = 100°$ (∠s between a line) $PBE = 40°$ $x + 40° + 40° = 180°$ (Sum of ∠s) ∠s on a straight $x = 180° - 80°$ $x = 100°$	Statement error Reasoning error
L07	$\hat{P} = 100°$ (given) $\hat{B}_3 = 40°$ (given) $\hat{B}_1 = 50°$ (tan-chord) $\hat{B}x = 180° - 40° - 50°$ (∠s on a straight line)	Statement error Reasoning error Last part misconception
L09	ABT is a tangent (it touches the circle at one point) ∴ $x = 100°$ (∠ between a line and a chord)	Reasoning error
L10	$\hat{B} = 90°$ [Tan ⊥ chord] $90° - 40° = 50°$ $x + 40° + 50° = 180°$ (∠s on a straight line $x + 90° = 180°$ $x = 90°$	Statement error (Operating below VHL1) Reasoning error (Operating below VHL1) Last part misconception

L11	AE = BT $\hat{E}$ = 40° (t [illegible] radii) $\hat{x}$ = 90 − 40° = 50°	Statement error (Operating below VHL1) Reasoning error (Operating below VHL1) Resulted in a misconception
L12	B = 90° (∠s on a chord) x = 95° (ext ∠s on of a triangle)	Statement error (Operating below VHL1) Reasoning error (Operating below VHL1) Resulted in a misconception
L13	y=40 (Tan chord) 100 + 40 + A$\hat{B}_2$P = 180 A$\hat{B}_2$ = 180 − 100 − 40 A$\hat{B}$P = 40 P$\hat{E}$B = 40 (sum ∠s OΔ) (∠s on str line) 180 − 40 − 4 = x x = 180 − 40 − 40 x = 100	(∠AB_2P) Angle naming error
L14	x = 100° [∠ between a line and a chord.	Reasoning error
L15	x = 100 (∠ on a str. line	Reasoning error

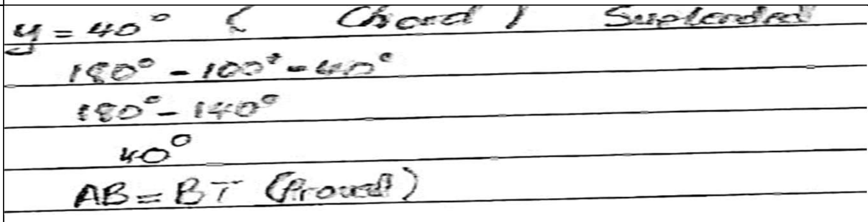
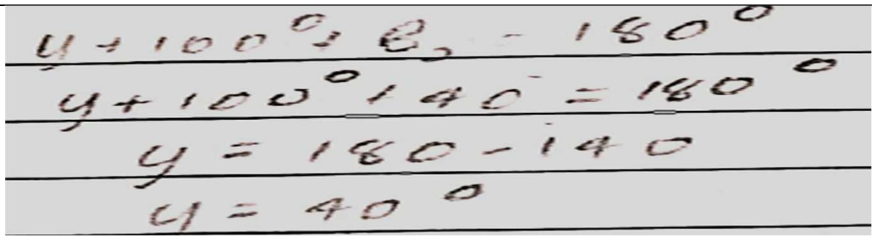
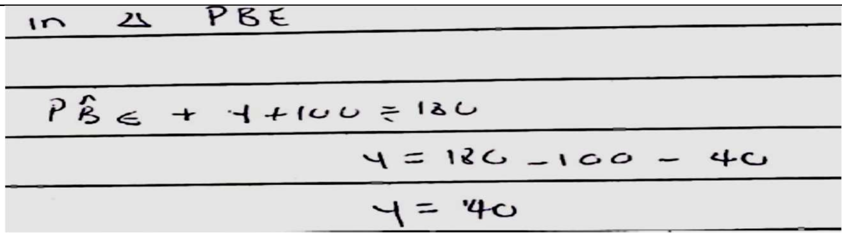
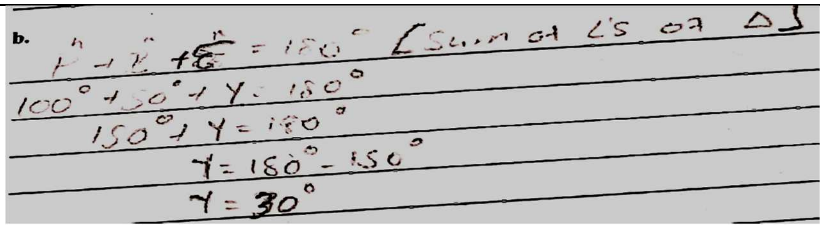
The Importance of the Diagram

The diagnostic report from the DBE underscores the importance of learners engaging meaningfully with diagrams. Based on L17's response, it appears that there was a lack of careful consideration of the diagram. To effectively apply the tangent chord theorem, learners must carefully analyze the diagram to identify tangents, chords, and their corresponding angles. Learners need to understand how angles are created at the tangent point and their connection to the angles formed by the chord.

Progress to Higher Levels of Reasoning

Based on the Van Hiele model, it appears that L17 is operating at a lower level, potentially at Level 2 (recognition), rather than progressing to Level 3 (informal deduction). To make progress, L17 needs targeted support to better connect diagram analysis with theorem application. This may entail engaging in activities that promote the exploration of the characteristics of tangents and chords through interactive geometry tasks.

Table 3: *Question item B learners from L16 to L30 responded with errors.*

Learner codes	Question Item B	Researchers' comments
L17		Reasoning error $AB = BT$ (Assumption/ Statement error)
L18		No Reasoning Assumption error
L22		No Reasoning
L23	NOT ATTEMPTED	No comment
L29		Assumption of Angle B Resulted in a misconception

DISCISSION

The study's findings indicate that many of the 30 participating learners showed a fundamental misunderstanding of the tangent-chord theorem, as evident in their written responses. Most learners were found to be functioning at Van Hiele levels 1 and 2, which indicates that they relied heavily on primarily visual and memorized procedures rather than deep conceptual understanding. Following an in-depth analysis of their responses and engaging in a comprehensive discussion, it can be deduced that a significant majority of the learners faced challenges when providing a correct geometric reason for the tangent-chord theorem task in geometry. This aligns with the study of Kpotosu et al. (2024), who found that many secondary school learners struggle with circle geometry, particularly when comprehending the critical relationship between the radius and tangent and the angle created by a tangent and a chord. Specifically, it was observed that 5 (16.7%) of the learners demonstrated a pure misunderstanding or had no knowledge of the concept at all. Additionally, 36.7% of the learners encountered statement errors on item A, while the same percentage encountered statement errors on item B. These errors clearly arise from carelessness (Aidoo-Bervell, 2021) and can be classified as either systematic or random

(Motseki & Luneta, 2024). It is evident that errors and misconceptions arose due to the learners' assumptions and their limited ability to visualize the geometric diagram, which ultimately hindered their progression beyond levels 1 and 2 of Van Hiele.

Consequently, in item A, only 12 learners provided both correct numerical values and appropriate reasons, with common errors including incorrect angle labeling and flawed assumptions. Many referenced unrelated theorems or used vague terminology. In item B, while 25 learners achieved the correct angle value, only 16 offered valid explanations, indicating a reliance on procedures rather than deep understanding, as seen with learner L17, who calculated correctly but relied on unrelated principles. As a result, it can be concluded that the learners did not attain a deep understanding of the fundamental concepts. This aligns with prior studies suggesting that learners at these levels struggle to link geometric theorems with diagram representations, as noted by Biber et al. (2013). Furthermore, tables 1 to 3 visually support these observations, indicating recurring challenges, including difficulty in recognizing tangent points, misuse of terminology, and weak diagram interpretation skills. Thematic analysis reveals that these challenges largely stem from poor visualization, overgeneralization of geometric rules, and instructional gaps in teaching diagrammatic reasoning, and these findings are consistent with the Van Hiele model. These results can serve as valuable feedback for educators, emphasizing the importance of focusing on learners' conceptual understanding in the development of mathematical knowledge.

CONCLUSION AND RECOMMENDATIONS

This study explored the errors and misconceptions encountered by 11th-grade learners while tackling problems associated with the tangent-chord theorem in circle geometry. Our findings revealed several interrelated challenges, the main among them being learners' difficulty in visualizing geometric relationships, limited understanding of mathematical symbols, and insufficient familiarity with geometric terminology (accepted geometric abbreviations), which hindered their ability to articulate their reasoning and engage with the material effectively. These obstacles also led many students to make unwarranted assumptions about the diagrams they encountered, resulting in misunderstandings in their problem-solving approaches. Lastly, their grasp of the various types of lines in a circle, along with their respective properties, was often inadequate, further complicating their understanding of the tangent-chord theorem. These factors collectively contributed to their difficulties in mastering this important geometric concept.

As a result, many learners were operating at or below level 1 (visualization) of the Van Hiele theory of geometric thinking than expected, far short of the expected reasoning level required to engage meaningfully with the tangent-chord theorem. To address these challenges, we recommend that mathematics teachers use diagnostic and reflective tests to identify learners' alternative geometric vocabulary and conceptual misunderstandings. Furthermore, incorporating dynamic geometry tools such as GeoGebra could help learners develop stronger visualization skills—a critical gap identified in this study. Additionally, professional development programs for teachers should focus on strategies for advancing learners through the Van Hiele levels, ensuring that they acquire not only procedural but also conceptual understanding of geometric theorems. Future research should explore the long-term impact of these interventions on learners' mathematical progression. Recognizing these misconceptions can inform the development of intervention strategies to improve learners' exploration of geometric concepts, language skills, and understanding of mathematical terms. The following recommendations are proposed to directly address identified gaps and improve geometry teaching:

Integrate visual learning tools

To address deficits in spatial reasoning and enhance learners' ability to visualize geometric relationships, dynamic geometry software such as GeoGebra should be systematically integrated into classroom teaching. These tools can help make abstract concepts tangible and foster intuitive geometric understanding.

Embed conceptual and open-ended tasks

Move beyond standard procedural tasks by incorporating open-ended questions that prompt learners to justify their mathematical reasoning and articulate their problem-solving processes.

These tasks require learners to explain their reasoning and justify their use of theorems, thereby promoting critical thinking and facilitating progression to Van Hiele upper levels.

Provide explicit instruction in geometry language and theorems

Teachers should place greater emphasis on the precise use of accepted geometric abbreviations or terminology, especially when dealing with tangents, chords, and angle relationships in circle geometry. Structured, language-focused instruction will improve learners' ability to articulate mathematical ideas and reason logically.

Conduct diagnostic and formative assessments

Implementing structured group work and peer teaching activities can enhance learners' metacognitive awareness and reinforce conceptual understanding. Encouraging learners to verbalize their thought processes helps clarify their reasoning and correct misunderstandings through social learning, ultimately promoting social constructivist learning.

Promote collaborative and peer learning

Create an environment that promotes peer-to-peer interaction and collaboration among learners. Encourage those who have a strong grasp of the concepts to explain and teach their peers, fostering a supportive learning community. This peer teaching approach can help clarify misconceptions, reinforce understanding, and build a strong foundation of knowledge through collaborative learning.

Contextualize learning with real-world applications

Introduce real-world problems that require the application of the tangent-chord theorem, demonstrating its relevance in practical scenarios. Linking mathematical concepts to real-world scenarios helps learners understand the theorem's importance and relevance in different practical situations. This contextualization of mathematical concepts can enrich learners' learning experiences and foster a more comprehensive understanding of the subject matter.

Ethical Consideration

The ethical considerations addressed in this research are briefly discussed in this section.

Permission

The researcher sought and received permission to conduct the research from the Tshwane University of Technology through the Ethical Clearance Office. Permission for the research was also obtained from the provincial Department of Education, the district office, and the school.

Informed consent

The participants were provided with a consent form. Signing the consent form indicated their agreement to take part in the study. They were informed that they could terminate their participation at any stage of the study without negative consequences.

Anonymity and Confidentiality

To protect participant confidentiality, pseudonyms were used in all documentation, and data was securely stored on password-protected devices. All participants were informed of their right to withdraw from the study at any time, and measures were taken to ensure that their participation did not interfere with their academic obligations.

Protection from harm

Participation in the research took place during normal school hours while the school security guard was present to ensure the safety of the participants. This ensured that no physical harm could be done to the participants. Moreover, the research did not harm the participants mentally or psychologically because the interview questions were aimed at protecting their dignity and rights.

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